Social Aggregation Without the Expected Utility Hypothesis

Charles Blackorby, David Donaldson, Philippe Mongin

ECOLE POLYTECHNIQUE

CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

SOCIAL AGGREGATION WITHOUT THE EXPECTED UTILITY HYPOTHESIS

Charles Blackorby
David Donaldson
Philippe Mongin

July 2004

Cahier n° 2004-020

LABORATOIRE D'ECONOMETRIE
1rue Descartes F-75005 Paris
(33) 1 55558215
http://ceco.polytechnique.fr/
mailto:labecox@poly.polytechnique.fr

SOCIAL AGGREGATION WITHOUT THE EXPECTED UTILITY HYPOTHESIS

Charles Blackorby
David Donaldson
Philippe Mongin

July 2004

Cahier n° 2004-020

Résumé: L'article examine les conditions qui permettent de satisfaire simultanément les versions ex ante et ex post du principe de Pareto, lorsqu'on cesse d'imposer l'hypothèse de l'utilité espérée aux préférences individuelles et sociales. Avec des probabilités subjectives qui peuvent varier, on obtient trois théorèmes d'impossibilité sans avoir à faire cette hypothèse. Lorsque les probabilités sont "objectives" (identiques pour tous), on obtient un théorème de caractérisation pour les préférences (Théorème 4). Celles des individus sont compatibles non seulement avec l'utilité espérée subjective, mais aussi avec certaines variantes non linéaires, notamment "l'utilité anticipée". Les préférences sociales sont en général du type séparable et pondéré. Ce résultat est à comparer au théorème d'agrégation de Harsanyi, qui part de l'hypothèse d'utilité espérée "objective" pour les individus et l'observateur.

Abstract: This paper investigates the possibilities for satisfaction of both the ex-ante and ex-post Pareto principles in a general model in which neither individual nor social preferences necessarily satisfy the Expected Utility Hypothesis. If probabilities are subjective and allowed to vary, three different impossibility results are presented. If probabilities are 'objective' (identical across individuals and the observer), necessary and sufficient conditions on individual and social value functions are found (Theorem 4). The resulting individual value functions are consistent not only with Subjective Expected Utility theory, but also with some versions of Prospect Theory, Subjectively Weighted Utility Theory, and Anticipated Utility Theory. Social Preferences are Weighted Generalized Utilitarian and, in the case in which individual preferences satisfy the Generalized Bernoulli Hypothesis, they are Weighted Utilitarian. The objective-probability results for social preferences cast a new light on Harsanyi's Social Aggregation Theorem, which assumes that both individual and social preferences satisfy the Expecte Utility Hypothesis.

Mots clés : Utilité espérée, utilité anticipée, séparabilité, principe de Pareto, économie du bien-être ex ante et ex post

Key Words : Expected Utility, Anticipated Utility, Separability, Pareto Principle, Ex-Ante and Ex-Post Welfare Economics

Classification JEL: D 60, D 69, D 81, D 71

Social Aggregation Without the Expected Utility Hypothesis*

Charles Blackorby,† David Donaldson,‡ and Philippe Mongin§

June 2004

Abstract

The paper investigates the consistency between the *ex ante* and *ex post* methods of social evaluation in a model of decision under uncertainty with very weak rationality assumptions. Neither the individuals nor the social observer are required to satisfy the expected utility hypothesis. In the case where individuals have the same subjective probabilities, Theorem 1 characterizes the nonlinear form of both individual and social utility. This result is best seen as a generalization of Harsanyi's theorem, the utilitarian conclusion of which is eschewed. In the case where subjective probabilities differ, Theorem 2 establishes an impossibility. This negative result extends earlier ones which depended on making the expected utility hypothesis. A welfare economics application of the theorems is offered.

1 Introduction

When a society is faced with the problem of ranking uncertain prospects, the computation can be made in two different ways depending on what set

*The first two authors gratefully acknowledge support from the Social Sciences and Humanities Research Council of Canada. The present version was completed when the third author was visiting the California Institute of Technology.

†Department of Economics, University of Warwick.

‡Department of Economics, University of Bristish Columbia.

§Corresponding author. Address: Laboratoire d'économétrie, Ecole Polytechnique, 1 rue Descartes, F-75005 Paris. E-mail: mongin@poly.polytechnique.fr

of preferences the Pareto principle is applied to. In the *ex ante* version, the principle bears on individual rankings of prospects, while in the *ex post* version, it is applied to individual rankings of final consequences. To each application corresponds a distinct method for constructing a social evaluation of prospects. One method, which is also called *ex ante*, directly aggregates the utility representations of individual preferences over prospects, and invokes the *ex ante* principle to make the aggregator Pareto-inclusive. In this approach, the individuals, but not necessarily the social aggregate, are required to obey some theory of decision under uncertainty; in practice, the latter has always been subjective expected utility theory (SEUT). The other method is more roundabout. For each state of the world, it aggregates the utility representations of individual preferences over consequences, invoking the *ex post* principle to make each state-relative aggregator Pareto-inclusive; then, it combines the resulting vector of utilities with some vector of probabilities, applying the same decision theory as for the individuals. Again, subjective expected utility theory (SEUT) has been the only one considered in practice. This approach is called *ex post* because the Pareto principle is applied in the *ex post* version, but it is no less *ex ante* than the other is; i.e., the computation is still made *before* the uncertainty is resolved. As regards to rationality considerations, the *ex post* approach makes a point of treating the individuals and social entity alike, a crucial difference with the *ex ante* approach.

Both at the level of principles and for practical applications, the question arises of which of the two versions is the more appropriate. Welfare theory has provided arguments one way or another rather than a final answer, and a glance at the literature in macroeconomics and general equilibrium theory shows that the matter is far from being settled there either. Take for instance the welfare analysis of money changes in the overlapping generation, rational expectation model initiated by Lucas (1972) [14]. Should the changes induced on second period consumption be evaluated in terms of the agents' first period expected utility functionals - an *ex ante* analysis? Or should they be evaluated in terms of the second period utility values for consumption in each state, combining these values with the central bank's probability values - an *ex post* analysis? Each line of analysis has much to say for itself. The former takes seriously the Samuelsonian dictum that "individual preferences count". Would they stop being relevant simply because the microeconomist's standard objects of preferences, i.e., allocations of commodity baskets, are replaced by something more complex, i.e., uncertain prospects over these objects? In favour of the latter approach, Hammond (1982) [10] has argued that it imposed expected utility rationality on the social aggregate, which the former does not do. It is also argued that the *ex post* method draws a clear separation between probabilistic and utility considerations, which the *ex ante* method somehow amalgamates.

An instructive, and indeed classic, way out of the theoretical dilemma is to explore the extent to which the two horns really differ, or relatedly, to investigate the logical consequences of assuming that they coincide. If it turned out that *ex post* - *ex ante* consistency would characterize a non-empty class of utility functions, it would appear to be normatively justified to evaluate policy changes in terms of that class, and no other; one would enjoy the best of two worlds, as it were. Which assumption provides the better descriptive fit is of course another matter, and it is not discussed here.

When *ex post* - *ex ante* consistency is combined with an expected utility definition of rationality, highly precise results can be obtained. Some of these results are positive (in the style of utilitarianism), while others are negative (in the style of the impossibilities of social choice theory). The theorems depend in part on the wording of the Pareto principle, i.e., on whether the weak, or strong, or the Pareto-indifference condition is selected. The present paper rejuvenates this general line of analysis by enlarging the framework of rationality. The assumptions we make are compatible not only with expected utility, but with a number of current generalizations. Our conclusions are again sometimes positive, sometimes negative, and at least for the negative part, not significantly weaker than those reached under expected utility, despite the weaker starting point.

To provide more perspective, we need briefly to discuss two earlier sets of results.[1] Harsanyi's (1955) [13] classic aggregation theorem assumes the *ex ante* Pareto principle in a context of lotteries and von Neumann-Morgenstern utility functions (for both the individuals and society). The conclusion is a weighted form of utilitarianism. To have a bearing on the previous discussion, Harsanyi's theorem must be rephrased as a result in SEUT, with Savagian acts (i.e. state-contingent functions) replacing Neumannian lotteries, and with the special assumption that all individuals share the same subjective probability measure. (This translation is fully

[1] Other references are important, but we make no claim to cover the field here. Mongin and d'Aspremont (1998) [18] provide an overview and complete bibliography.

developped in Blackorby, Donaldson, and Weymark (1999) [2].) Within this framework, *ex post - ex ante* consistency is automatically ensured. Technically, Harsanyi's initial statement relied on the Pareto-indifference condition, but later variants have invoked the strong Pareto condition in order to sign the weights positively in the utilitarian sum.

Our positive result, Theorem 1, is best seen to be a strengthening of Harsanyi's, in its SEUT rephrasing and its technical variant based on the strong Pareto principle. We endow the individuals and the bearer of social evaluation with one and the same subjective probability, and since our decision-theoretic assumptions are weak, *ex post - ex ante* consistency delivers a social utility function that is not utilitarian, but only additively separable in individual utilities. This conclusion sounds like good news for those welfare economists who would like to evaluate policy changes in terms of a consistent social aggregate function, but are reluctant to fall back onto Bentham's and Jevons's formula. Also, and more importantly, Theorem 1 characterizes the form of individual and social value functions, and the latter turn out to come very close to standard expected utility functions. Here, the added value of the theorem lies in the fact that it *deduces* from the antecedent consistency condition what Harsanyi bluntly puts in his assumptions - and arguably, it *justifies* the latter by the former.[2]

There is a definite analogy between this component of Theorem 1 and the work pursued on dynamic consistency in the single-individual case, which also accounts for expected utility in terms of antecedent consistency conditions. However, *ex post-ex ante* consistency, as defined here, is not equivalent to any of the conditions in Hammond's (1988) classic decomposition of expected utility, and furthermore, we impose it only at the *collective* level. As the theorem shows, no more than this collective rationality condition is needed in order to turn individual rationality into (what comes close to) expected-utility maximization.

Harsanyi's characterization of weighted utilitarianism breaks down when SEUT is given its full force, i.e., when subjective probabilities are allowed to differ. This is demonstrated by Mongin (1995) [17] in a Savage model of acts and preferences that accommodates as much variability or uniformity in utilities and probabilities as one wishes to assume. Weighted utilitarian-

[2] By the same token, Theorem 1 is stronger than Myerson's (1981) [19] variant of Harsanyi's theorem. The latter imposes *ex post-ex ante* consistency on the collective preference, but unlike our theorem, retains the expected utility assumption on the individuals' side.

ism is seen to correspond to one polar case (identical probabilities), whereas a dual weighted "probabilitarian" theorem holds for the other polar case (identical utilities). In the territory between the two poles, impossibility conclusions crop up. Depending on the technical variant of the Pareto principle, they range from quasi-Arrovian dictatorships to sheer logical impossibilities; even the weak Pareto-indifference condition creates trouble. The upshot of this analysis is that *ex post - ex ante* consistency is incompatible with diversity of beliefs and evaluations, as long as the rationality conditions are those of expected utility theory.

Theorem 2 of this paper allows subjective probabilities to differ, and derives a logical impossibility analogous to Mongin's result for the strong Pareto principle (we do not try to cover the other variants of the principe). Some of our assumptions do not have counterparts among Savage's, which makes the comparison with the earlier paper imperfect. However, from a broader point, Theorem 2 makes it clear that the impossibility of "consistent Bayesian aggregation" is not specific to "Bayesianism". The practical implication is that to exempt society from the sure-thing principle does not provide a way out, contrary to what is often suggested in response to the logical conflict. What is left from the initial SEUT model being so weak as to be virtually unassailable, one has to face the blunt choice between the *ex ante* and *ex post* method, and their respective philosophies of the Pareto principle.

Both Theorems 1 and 2 may be compared with the work most recently pursued on social preference under risk and uncertainty. A feature of this work that is present here is to relax the expected utility assumptions, which for a long time provided the only modelling of risk and uncertainty for the collective context. However, more often than not, the other writers derive, and normatively defend, nonseparable preference rules for the social evaluation. Besides, to the best of our knowledge, none of them has yet followed the strategy adopted here of inferring the form of individual evaluations from prior, normative assumptions put at the collective level.[3]

The emphasis of this paper is not on axiomatizations. We deliberately made utility and probability evaluations, rather than preferences, the primitive terms of the analysis. The utility side is described by numbers rather

[3] As a brief and incomplete sample of the work pursued along this line, we may cite Epstein and Segal (1992) [5], Karni and Safra (2002) [12], as well as, for the related topic of inequality measurement, Ben Porath, Gilboa, and Schmeidler (1997) [1]

than functions, the Paretian constraints being expressed directly in terms of these derived quantities. In one sense, we are assuming "welfarism", though in another sense, we are not, since we take note of probabilities (beliefs) besides utilities (welfare).[4] Such a compressed mode of presentation makes it possible to reach conclusions quickly. We make the simplifying assumption that, at least over the range relevant to social decisions, each individual's utility values vary independently of the utility values reached by the others. This situation is automatically realized in the privileged application of the paper, which stems from standard welfare economics.

2 Assumptions

2.1 Individuals

Let $N = \{1, ..., n\}$ be the set of individuals and $M = \{1, ..., m\}$ the set of states of nature. Throughout, we require that individual i evaluate prospects with a value function $v^i = V^i(\mathbf{p}^i, \mathbf{u}^i)$, where $\mathbf{p}^i = (p_1^i, ..., p_m^i)$, $\mathbf{u}^i = (u_1^i, ..., u_m^i)$, and p_j^i is the probability value that i assigns to state j, u_j^i is the utility value that i achieves in state j. When utility values are listed by the states, they will be denoted $\mathbf{u}_j = (u_j^1, ..., u_j^n)$, $j \in M$.

In the value function V^i, both $\mathbf{p}^i$ and $\mathbf{u}^i$ are allowed to vary; however, the two variables will play a very different rôle. Concerning the $\mathbf{p}^i$, we require only two things, i.e., that they can take any values in the unit-simplex $\mathcal{S}_1^M$ of R^M, and that whatever assumption is made on $\mathbf{u}^i$ holds irrespectively of the value taken by $\mathbf{p}^i$. Granted this, the assumptions amount to putting special constraints on the $\mathbf{u}^i$, which are the active variables in the model. By and large, this model conforms with Savage's, in which probability measures are defined on states of nature, but there are technical dissimilarities that we do not pursue.[5] The case of identical probability vectors, $\mathbf{p}^1 = ... = \mathbf{p}^n = \mathbf{p}$, will be dealt with in section 3, and the other case in section 4.

For each i, we will assume that the values u_j^i belong to a non-degenerate

[4] *Welfarism* is the claim that the information relevant to the social evaluation is well-reflected in the individuals' utility values, so that the physical differences between objects of choice become irrelevant.

[5] Savage's (1972) [20] construction allows for variability in the probability and utility values only implicitly, i.e., through the assumption of a rich set of acts and the divisibility postulate (P6), whereas we allow for it explicitly here. This is the major difference.

interval[6] that is independent of the state j:

Assumption 1 : *For all* $i \in N$, $j \in M$, $u_j^i \in \mathcal{U}^i$, *a non-degenerate interval.*

Expected utility (EU) as well as several non-expected utility (NEU) functionals have their range of values equal to an interval. For these theories, the only problematic part of the assumption is whether the ranges of values reached in a particular state have a significant intersection with the ranges in the other states. Both in the EU and NEU case, difficulties may occur when the utility functions are *state-dependent*, i.e., have the form $u^i(j,.)$. For instance, if j is the state in which i is alive and k the state in which i is dead, it may be right to assume that $u^i(j,.)$ and $u^i(k,.)$ have no values in common. However, this example is somewhat extreme. In case of intuitively less distant states of affairs, it seems acceptable to assume the intersection property; e.g., take j to be the state in which the weather is hot and k the state in which it is cool, and suppose the variable x in $u^i(j,x)$ and $u^i(k,x)$ ranges over allocations of beercans. Note carefully that the assumption does not require the *functions* $u^i(j,.)$ to be identical or even to determine the same ordering, and it does not require the ranges of the $u^i(j,.)$ to be identical outside the subset of interest $\mathcal{U}^i$.

For any function f, *Rge* f denotes its range of values, and for any number x, $\bar{\mathbf{x}}$ denotes the constant vector $(x, ..., x)$. Using this symbolism, we impose two restrictions on the values v^i taken by the functions V^i. Informally, the first restriction states that putting u^i as a utility input uniformly across states gives back u^i, whatever the probability vector $\mathbf{p}^i$, and the second, that the same values u^i can be used as utility inputs in any state:

Assumption 2.1 : *For all* $i \in N$, $\mathbf{p}^i \in \mathcal{S}_1^M$, *and* $u^i \in \mathcal{U}^i$, $V^i(\mathbf{p}^i, \bar{\mathbf{u}}^i) = u^i$.

Assumption 2.2 : *For all* $i \in N$, *Rge* $V^i \subseteq \mathcal{U}^i$.

Together, the two assumptions set *Rge* $V^i = \mathcal{U}^i$. Given the utility functions standardly used in economics, neither assumption is very demanding. As to 2.1, it is trivially satisfied by EU, and after suitable normalization, by several NEU functionals. As to 2.2, consider $\sum p_j^i u^i(x_j)$, where $x_j \in X$ is the social alternative realized in state j, X is a connected space, and u^i is continuous. Then, the range of $\sum p_j^i u^i(x_j)$ is an interval, and by adding the assumption that u^i is increasing (in the sense of respecting the preference

[6] A non-degenerate interval is not reduced to a point; an interval may be bounded or unbounded.

ordering assumed on X), we make this range exactly equal to that of u^i.[7] This is a familiar argument in EUT. To extend it to NEU, we need to postulate the continuity and monotonicity of the V^i explicitly, which is done in the next assumption. Notice that state-dependence does not create an additional problem here. It does not matter whether the u^i_j come from a representation $u^i(x_j)$ or $u^i(j, x_j)$, provided these values come from $\mathcal{U}^i$.

We make another assumption that EU and other representations trivially satisfy.

Assumption 3 : *For all* $i \in N$ *and* $\mathbf{p}^i \in \mathcal{S}_1^M$, $V^i(\mathbf{p}^i, .)$ *is continuous and increasing.*

2.2 The social aggregate

To state our second group of assumptions, we introduce the function

$$\varphi(\mathbf{p}^1, ..., \mathbf{p}^n, \mathbf{u}^1, ..., \mathbf{u}^n),$$

which indicates how the social evaluation varies with the set of individual data. To make sense of this description, we assume that the order of variables in φ does not matter when the indexes are kept unchanged. Here comes a more substantial assumption.

Assumption 4. *The domain of* φ is *the product set*:

$$\mathcal{D} = (\underbrace{S_1^M \times ... \times S_1^M}_{n \text{ times}}) \times (\underbrace{\mathcal{U}^1 \times ... \times \mathcal{U}^1}_{m\ times}) \times ... \times (\underbrace{\mathcal{U}^n \times ... \times \mathcal{U}^n}_{m\ times}) .$$

This assumption says that probability and utility values can vary fully and independently of each other. The Cartesian product domain for the utilities has the obvious defect of ignoring the psychological interdependencies that may prevail between the individuals. It excludes, say, that Daesdemona will reach her bliss point only when Othello reaches his, and Iago enjoys himself most when Othello is at his worst. However, our intended applications are more down-to-earth. The following example is canonical for the whole paper.

[7] "Increasing" in this paper, means what others call "strictly increasing"; otherwise, we say "weakly increasing". For multivariate functions, "increasing" will mean "increasing in each argument separately".

Example 1. Take r to represent the number of commodities in the economy and the elements of $X = R_+^{rn}$ to represent all possible allocations of baskets of commodities between the n individuals. Suppose that i evaluates the prospect that for $j \in M$, $x_j \in X$ will occur by employing the state-independent EU functional $\sum p_j^i u^i(x_j)$. If $u^i(.)$ depends only on i's basket of commodities, is continuous and increasing, take $\mathcal{D}$ to be the set of all utility allocations across states and individuals. The restrictions made on the $u^i(.)$ are those of standard welfare economics *à la* Bergson-Samuelson. They are taken up in Hammond's (1981) [9] early discussion of the *ex post* - *ex ante* problem. They can be weakened by considering state-dependent functions $u^i(j, x_j)$ that satisfy Assumptions 1 and 2.1.

Without loss of generality, the φ function can be written as a value function V^0 similar to the value functions V^i of the individuals (and henceforth, we will refer to the social entity as if it were an added individual, "the social observer"). That is to say, there exist a function V^0, a probability vector function $\mathbf{p}^0$, and a utility vector $\mathbf{u}^0$, such that for all i, and $\mathbf{p}^i$ and $\mathbf{u}^i$ in the domain,

$$\varphi(\mathbf{p}^1, ..., \mathbf{p}^n, \mathbf{u}^1, ..., \mathbf{u}^n) = V^0(\mathbf{p}^0(\mathbf{p}^1, ..., \mathbf{p}^n, \mathbf{u}^1, ..., \mathbf{u}^n), \mathbf{u}^0(\mathbf{p}^1, ..., \mathbf{p}^n, \mathbf{u}^1, ..., \mathbf{u}^n)).$$

This formal rephrasing is permissible because it makes the social probability and social utility vectors depend on the *whole* collection of individual data. Now, the remaining constraints can be stated in terms of $\mathbf{p}^0, \mathbf{u}^0$, and V^0.

We have no plausible constraint to put on the social probability vector function $\mathbf{p}^0$, except that it should not depend on the utilities. For the purpose of the results to come, there is no loss in taking it to be constant. Concerning the social utility vector function $\mathbf{u}^0$, we adopt the *ex post* approach to social preference by making the observer's utility in state j an increasing function of individual utilities in that state. We strengthen the constraint by stipulating that the u_j^i be related to u_j^0 by a *state-independent* function U^0. This restriction is usually taken for granted in the *ex post* approach.

Assumption 5: *There is a fixed probability vector* $\mathbf{p}^0$, *and there are increasing functions* U^0 *and* $V^0(\mathbf{p}^0, .)$ *s.t. for all* $(\mathbf{p}^1, ..., \mathbf{p}^n, \mathbf{u}^1, ..., \mathbf{u}^n) \in \mathcal{D}$,

$$\varphi(\mathbf{p}^1, ..., \mathbf{p}^n, \mathbf{u}^1, ..., \mathbf{u}^n) = V^0(\mathbf{p}^0, U^0(\mathbf{u}_1), ..., U^0(\mathbf{u}_m)).$$

Concerning V^0, we adopt the *ex ante* approach to social preference. If the decision-theoretic construction were made explicit, the following assumption would amount to applying the Strong Pareto Principle to individual preferences over prospects.

Assumption 6 : *There is an increasing function* W *s.t.*

$$\textit{for all } (\mathbf{p}^1, ..., \mathbf{p}^n, \mathbf{u}^1, ..., \mathbf{u}^n) \in \mathcal{D},$$

$$\varphi(\mathbf{p}^1, ..., \mathbf{p}^n, \mathbf{u}^1, ..., \mathbf{u}^n) = W(V^1(\mathbf{p}^1, \mathbf{u}^1), ..., V^n(\mathbf{p}^n, \mathbf{u}^n)).$$

In our reduced form model, the increasing property of W and U^0 echoes the strong Pareto condition, and the very existence of these functions, the Pareto-indifference condition. So we have taken aboard the strong Pareto principle, in both the *ex ante* and *ex post* way. Combining the last two assumptions, we get the consistency condition that the *ex ante* and *ex post* modes of social aggregation agree with each other.

We finally need a technical analogue of Assumption 3:

Assumption 7 : V^0, U^0 *and* W *are continuous functions.*

Note that the same decision-theoretic constraints have been imposed on the social observer and the individuals, which is in keeping with the *ex post* approach. To make this analogy clearer, we have refrained from cancelling the slight redundancies between the conditions.

3 Identical Subjective Probabilities: A Characterization Theorem.

This section characterizes the observer's and individuals' preferences, given Assumptions 1 to 7 and the special condition that all probability vectors, including the observer's, are equal to some common $\mathbf{p}$, which can take any values. Packing Assumptions 4-6 into a single one, we get the condition that for all $i \in N$, $j \in M$, $u_j^i \in \mathcal{U}^i$, and $\mathbf{p} \in \mathcal{S}_1^M$,

$(*)$ $V^0(\mathbf{p}, U^0(\mathbf{u}_1), ..., U^0(\mathbf{u}_m)) = \varphi(\mathbf{p}, ..., \mathbf{p}, \mathbf{u}^1, ..., \mathbf{u}^n)$
$= W(V^1(\mathbf{p}, \mathbf{u}^1), ..., V^n(\mathbf{p}, \mathbf{u}^n))$.

We will refer to condition $(*)$ as to *ex post - ex ante consistency.*

Theorem 1 *Suppose that Assumptions 1-4 hold. Then, Assumptions 5-7 hold if and only if there are continuous and increasing transformations* g^0,

$g^1, ..., g^n$, w^0 and v^0, and continuous and positive functions $a_1(\mathbf{p}), ..., a_m(\mathbf{p})$, such that for all $(\mathbf{p}^1, ..., \mathbf{p}^n, \mathbf{u}^1, ..., \mathbf{u}^n)$ in $\mathcal{D}$,

$$U^0(u_j^1, ..., u_j^n) = g^0(\sum_{i \in N} g^i(u_j^i)), \; j \in M, \tag{1}$$

$$V^i(\mathbf{p}, \mathbf{u}^i) = (g^i)^{-1}(\sum_{j \in M} a_j(\mathbf{p}) g^i(u_j^i)), \; i \in N, \tag{2}$$

and:

$$W(V^1(\mathbf{p}, \mathbf{u}^1), ..., V^n(\mathbf{p}, \mathbf{u}^n)) = w^0(\sum_{i \in N} g^i(V^i(\mathbf{p}, \mathbf{u}^i)), \tag{3}$$

$$V^0(\mathbf{p}, U^0(\mathbf{u}_1), ..., U^0(\mathbf{u}_m)) = v^0 \circ (g^0)^{-1}(\sum_{j \in M} a_j(\mathbf{p}) g^0(u_j^0)). \tag{4}$$

The proof of the theorem is technical; we state it in the Appendix.

Generally, a multivariate function $f = f(x_1, ..., x_Q)$ is said to be *additively separable* if there exist functions $f_1, ..., f_Q$ and an increasing function g such that

$$f(x_1, ..., x_Q) = g(f_1(x_1) + ... + f_Q(x_Q)).)$$

Conclusions (1) and (2) state that social utility is additively separable in the individual utilities. This functional form is a generalization of, and arguably an improvement on, Harsanyi's (1955) weighted utilitarian sum. Our formula still permits compensating i's decrease of utility with j's increase without changing the other individuals' utilities, but it eschews the dubious linearity in the marginal compensation that is postulated by utilitarians.

The decision-theoretic conclusions (3) and (4) say that the value functions must be additively separable across states, with each term being a product of a probability-relative weight with a utility-relative factor. They also say that the probability-relative weights $a_j(\mathbf{p})$ must be the same for all i. The exact connection between EU and the present functional form is mathematically unclear. It is known that in the presence of First Order Stochastic Dominance, the form $\sum_{j \in M} f^i(p_j, u_j^i(x_j))$ collapses into EU (see Machina, 1981, section 4.c.1). First Order Stochastic Dominance would be a natural assumption to add, and it will indeed be added below, but even so, it must be pointed out that $\mathbf{p}$, not just p_j, enters the weight of $u_j^i(x_j)$. *Practically, if not mathematically*, (3) and (4) do not go beyond EU. This conclusion follows from considering the NEU functions commonly used today. In Machina's (1991) synthetic list, there appear to be only two candidates. On further inspection, one of them does not satisfy (3) and (4), while the other one does; regrettably, it is the less representative of the two which satisfies the conditions.[8]

Example 2. Take the *rank-dependent utility* (RDU) formula:

$$\begin{aligned}&\left[1 - f^i(p_{j_2} + ... + p_{j_m})\right] \cdot \; u^i(x_{j_1}) \\ &+ \left[f^i(p_{j_2} + ... + p_{j_m}) - f^i(p_{j_3} + ... + p_{j_m})\right] \cdot (u^i(x_{j_2}) \\ &+ ... + f^i(p_m) \cdot u^i(x_{j_m}).\end{aligned} \tag{5}$$

Here the consequences x_j have been reordered in an increasing order of preference, $x_{j_1} \prec_i ... \prec_i x_{j_M}$ (for simplicity, we state the formula in the no-indifference case), and the probability-distorsion function $f^i : [0,1] \to [0,1]$ is increasing, continuous, and normalized by $f^i(0) = 0$, $f^i(1) = 1$. With this standard normalization, Assumption 2.1 is met. When $u^i(.)$ is continuous and increasing, Assumptions 1-3 hold. However, RDU is *not* an example of (3) or (4), because the weight it assigns to u^i_j depends on the rank-order of u^i_j *vis-à-vis* the other components u^i_k. That is to say, in the case of RDU, the weights obey the form $a_j(\mathbf{p}, \mathbf{u}^i)$, not $a_j(\mathbf{p})$ as required. Most of the available NEU representations, including the differentiable forms that treat the EU formula as a local linear approximation, are of the type $a_j(\mathbf{p}, \mathbf{u}^i)$, hence not compatible with the conclusions.

Example 3. The following functions have also been proposed:

$$\sum_{j \in M} g^i(p_j) u^i_j(x_j) \tag{6}$$

and

$$\sum_{j \in M} \frac{h^i(p_j)}{\sum_k h^i(p_k)} u^i_j(x_j), \tag{7}$$

with $g, h \geq 0$. Given the previous assumptions on $u^i_j(x_j)$, the former example (*prospect theory*) satisfies Assumptions 1-3, once the normalization $\sum_{j \in M} g^i(p_j) = 1$ is made, and so does the latter, which already includes

[8]It is a matter of dispute who "invented" examples 2 and 3, so we prefer not to make any attribution. Machina's (1991) states the relevant bibliographical references.

the normalization. As the utilities play no role in determining the weights, these two representations straightforwardly agree with the functional form in (3) and (4). However, the departure they allow from expected utility is minimal. The weights only have the effect of changing the initial probability vector $\mathbf{p}$ into some $\mathbf{p}'$, and by slightly strenghtening the assumptions, it would be possible to conclude that $\mathbf{p} = \mathbf{p}'$. Example 3 does not amount to a true generalization of EU theory.[9]

Given the insights provided by these two examples, there appears to be little use for formulas (3) and (4) beyond standard expected utility theory. Practically, the initial flexibility permitted by the weak rationality assumptions disappears from the conclusions. Theorem 1 is no less significant for that. It is striking that the EU-like conditions are now found *in the conclusions*, whereas Harsanyi put them in the assumptions; apparently, he did not realize the logical power of his own framework. In sum, Harsanyi was wrong both to *conclude* utilitarianism (instead of the weaker additive separability) and to *assume* expected utility (instead of just *ex post* - *ex ante* consistency). However, this is just our interpretation, and others might see Theorem 1 not as an improvement on Harsanyi's positive result, but rather as an impossibility theorem of a novel sort. As Gorman (1987) [8] wrote, "That a set of axioms implies additivity is as likely to be evidence against them as for it". If separability across separability in the states is regarded as normatively undesirable by some, our result will provide them with a *reductio* of *ex post* - *ex ante* consistency.

4 Different Subjective Probabilities: An Impossibility Theorem

By giving up the assumption that the individuals' and social observer's probabilities are identical, we derive a logical inconsistency. Contrary to the positive theorem of last section, this impossibility theorem requires

[9]We show that $\mathbf{p} = \mathbf{p}'$ by adding the following local variant of Assumption 2.2: for any two states j_1 and j_2, if $u^i_{j_1} = u^i_{j_2}$, then $V^i(\mathbf{p}^i, \mathbf{u}^i) = V^i(\mathbf{q}^i, \mathbf{u}^i)$ for all probability vectors $\mathbf{p}^i, \mathbf{q}^i$ satisfying $p^i_{j_1} + p^i_{j_2} = q^i_{j_1} + q^i_{j_2}$, and $p^i_k = q^i_k, k \neq j_1, j_2$. A straightforward functional equation argument then leads to the conclusion.

variability only in the utility values, so we will from now on regard the probability vectors $\mathbf{p}^1, ..., \mathbf{p}^n$ as being fixed. We need a decision-theoretic assumption not yet introduced, i.e., that if state k has a higher probability than state l, then i prefers to have the higher utility amount put on k than l. For transparent reasons, we call this requirement the *betting property for* i. In a context where the probability vectors $\mathbf{p}^i$ are allowed to vary, it would follow from preference for first-order stochastic dominance. Both EU and established NEU theories, such as RDU, trivially satisfy it.

Assumption 8.1. *For* $i \in M$, *suppose that* $p^i_k > p^i_l$ *for some* $k, l \in M$. *Then, for all utility values* u^i_k, u^i_l, u *such that* $u^i_k > u^i_l$ *and* $u^i_j = u$, $j \neq k, l$,

$$V^i(\mathbf{p}^i, u, ..., u^i_k, ..., u^i_l, ..., u) > V^i(\mathbf{p}^i, u, ..., u^i_l, ..., u^i_k, ..., u).$$

Assumption 8.2 *imposes the same condition on* $i = 0$.

Theorem 2 *Suppose that Assumptions 1 and 4 are satisfied with fixed* $\mathbf{p}^0$,$\mathbf{p}^1, ..., \mathbf{p}^n$. *Then,*

1. *If* $\mathbf{p}^i \neq \mathbf{p}^h$ *for individuals* $i, h \in N$, *there is no social value function* φ *that satifies Assumptions 5, 6 and 7, and 8.1.*

2. *If there exists* $i \in N$ *such that* $\mathbf{p}^i \neq \mathbf{p}^0$, *there is no social value function* φ *that satisfies Assumptions 5, 6, 7, 8.1 and 8.2.*

Proof of the theorem. (i) Let us assume that two individuals i and h have distinct probability vectors and derive a contradiction. Without loss of generality, let $i = 1, h = 2$, $p^1_1 > p^1_2$, and $p^2_1 < p^2_2$. Fix any $(\xi_1, \xi_2, \ldots, \xi_n)$ in the interior of $\times_{i\in N}\mathcal{U}_i$. From Assumption 5, there are η_1, $\eta_2 > 0$ s. t.

$$\begin{aligned} U^0(\xi_1 + \eta_1, \xi_2, \ldots, \xi_n) &> U^0(\xi_1, \xi_2, \ldots, \xi_n), \\ U^0(\xi_1, \xi_2 + \eta_2, \ldots, \xi_n) &> U^0(\xi_1, \xi_2, \ldots, \xi_n). \end{aligned} \quad (8)$$

Take $\bar{y}$ to be the smaller value on the left of (8) and $\bar{u}^0$ to be such that $\bar{y} > \bar{u}^0 > U^0(\xi^1, \xi^2, \ldots, \xi^n)$. Then, Assumption 5 implies that there are $\epsilon_1, \epsilon_2 > 0$ to satisfy the equation:

$$\bar{u}^0 = U^0(\xi_1 + \epsilon_1, \xi_2, \ldots, \xi_n) = U^0(\xi_1, \xi_2 + \epsilon_2, \ldots, \xi_n). \quad (9)$$

Consider first the vector $(\mathbf{p}^1, \tilde{\mathbf{u}}^1, \ldots, \mathbf{p}^n, \tilde{\mathbf{u}}^n)$ with

$$\begin{aligned} \tilde{\mathbf{u}}^1 &= (\xi_1 + \epsilon_1, \xi_1, \ldots, \xi_1) \\ \tilde{\mathbf{u}}^2 &= (\xi_2, \xi_2 + \epsilon_2, \ldots, \xi_2) \end{aligned} \tag{10}$$

and for $i \neq 1, 2$,

$$\tilde{\mathbf{u}}^i = (\xi_i, \ldots, \xi_i). \tag{11}$$

Consider second the vector $(\mathbf{p}^1, \check{\mathbf{u}}^1, \ldots, \mathbf{p}^n, \check{\mathbf{u}}^n)$

$$\begin{aligned} \check{\mathbf{u}}^1 &= (\xi_1, \xi_1 + \epsilon_1, \ldots, \xi_1) \\ \check{\mathbf{u}}^2 &= (\xi_2 + \epsilon_2, \xi_2, \ldots, \xi_2) \end{aligned} \tag{12}$$

and for $i \neq 1, 2$,

$$\check{\mathbf{u}}^i = (\xi_i, \ldots, \xi_i). \tag{13}$$

Assumption 8.1 implies that

$$V^1(\mathbf{p}^1, \xi_1 + \epsilon_1, \xi_1, \ldots, \xi_1) > V^1(\mathbf{p}^1, \xi_1, \xi_1, +\epsilon_1, \ldots, \xi_1)$$

and

$$V^2(\mathbf{p}^2, \xi_2, \xi_2 + \epsilon_2, \xi_2, \ldots, \xi_2) > V^2(\mathbf{p}^2, \xi_2 + \epsilon_2, \xi_2, \ldots, \xi_2),$$

so that Assumption 6 yields

$$\varphi(\mathbf{p}^1, \tilde{\mathbf{u}}^1, \ldots, \mathbf{p}^n, \tilde{\mathbf{u}}^n) > \varphi(\mathbf{p}^1, \check{\mathbf{u}}^1, \ldots, \mathbf{p}^n, \check{\mathbf{u}}^n). \tag{14}$$

However, Assumption 5 and equation (9) imply that :

$$\begin{aligned} &\varphi(\mathbf{p}^1, \tilde{\mathbf{u}}^1, \ldots, \mathbf{p}^n, \tilde{\mathbf{u}}^n) \\ &= V^0(\mathbf{p}^0, U^0(\xi_1{+}\epsilon_1, \xi_2, \ldots, \xi_n), U^0(\xi_1, \xi_2{+}\epsilon_2, \ldots, \xi_n), \ldots, U^0(\xi_1, \xi_2, \ldots, \xi_n)) \\ &= V^0(\mathbf{p}^0, U^0(\xi_1, \xi_2{+}\epsilon_2, \ldots, \xi_n), U^0(\xi_1, +\epsilon_1, \xi_2, \ldots, \xi_n), \ldots, U^0(\xi_1, \xi_2, \ldots, \xi_n)) \end{aligned}$$

$$= \varphi(\mathbf{p}^1, \check{\mathbf{u}}^1, \ldots, \mathbf{p}^n, \check{\mathbf{u}}^n). \tag{15}$$

a contradiction with (14). This establishes the first part.

(2) Now, suppose that $\mathbf{p}^i \neq \mathbf{p}^0$ for some $i \in N$. Given the first part, we need only consider the case in which $\mathbf{p}^1 = \mathbf{p}^2 = \ldots = \mathbf{p}^n$. Let $p^1_1 > p^1_2$ and $p^0_1 < p^0_2$ and consider $(\mathbf{p}^1, \tilde{\mathbf{u}}^1, \ldots, \mathbf{p}^n, \tilde{\mathbf{u}}^n)$ and $(\mathbf{p}^1, \check{\mathbf{u}}^1, \ldots, \mathbf{p}^n, \check{\mathbf{u}}^n)$ with $\tilde{\mathbf{u}}^1$ and $\check{\mathbf{u}}^1$ as before and

$$\tilde{\mathbf{u}}^i = \check{\mathbf{u}}^i = (\xi_i, \ldots, \xi_i) \tag{16}$$

for $2 \leq i \leq n$. Assumptions 6 and 8.1 imply that

$$\begin{aligned}
& \varphi(\mathbf{p}^1, \tilde{\mathbf{u}}^1, \ldots, \mathbf{p}^n, \tilde{\mathbf{u}}^n) \\
= \; & W(V^1(\mathbf{p}^1, \xi_1, +\epsilon_1, \xi_1, \ldots, \xi_1), \ldots, V^i(\mathbf{p}^i, \xi_i \ldots, \xi_i), \ldots) \\
> \; & W(V^1(\mathbf{p}^1, \xi_1, \xi_1 + \epsilon_1, \xi_1, \ldots, \xi_1), \ldots, V^i(\mathbf{p}^i, \xi_i \ldots, \xi_i), \ldots) \\
= \; & \varphi(\mathbf{p}^1, \check{\mathbf{u}}^1, \ldots, \mathbf{p}^n, \check{\mathbf{u}}^n). \qquad (17)
\end{aligned}$$

However, Assumptions 5 and 8.2 imply that

$$\begin{aligned}
& \varphi(\mathbf{p}^1, \tilde{\mathbf{u}}^1, \ldots, \mathbf{p}^n, \tilde{\mathbf{u}}^n) = \\
& V^0(\mathbf{p}^0, U^0(\xi_1 + \epsilon_1, \xi_2, \ldots, \xi_n), U^0(\xi_1, \xi_2, \ldots, \xi_n), U_0(\xi_1, \xi_2, \ldots, \xi_n), \ldots) \\
& < V^0(\mathbf{p}^0, U^0(\xi_1, \xi_2, \ldots, \xi_n), U^0(\xi_1 + \epsilon_1, \xi_2, \ldots, \xi_n), U^0(\xi_1, \xi_2, \ldots, \xi_n), \ldots) \\
& \qquad = \varphi(\mathbf{p}^1, \check{\mathbf{u}}^1, \ldots, \mathbf{p}^n, \check{\mathbf{u}}^n). \qquad (18)
\end{aligned}$$

This contradicts (17), establishing the second part.

Theorem 2 says that under the assumptions, the individuals' and the observer's probabilities must be the same if one is to avoid a contradiction. It is related to a negative result reached by Hammond (1981) in the context of the welfare economics discussion. Hammond's argument depends on making EU assumptions and differentiability assumptions, and we show here that neither assumption is necessary. Mongin (1995) covers a wider number of Pareto cases than we do here, but even a comparison restricted to the strong Pareto principle is difficult to pursue in detail, because the variability needed for his theorems is always found on the probability side, not on the utility side as in the present theorem.

Disregarding axiomatic dissimilarities, a interesting feature emerges from the proof of Theorem 2 when it is compared with related arguments.[10] The argument relied not only on applying the *ex ante* and *ex post* versions of the Pareto principle in succession, but also, and less obviously, in applying the *ex ante* principle in two different ways. In the second part of the proof, it was applied to the intuitively natural context in which unanimity in comparing prospects is rooted in a prior probabilistic agreement between the individuals. In the first part, however, it was applied to the less natural context in which two individuals disagreed in *both* their probability and

[10] Example 3 in Mongin (1995) highlights this feature. It is more subdued in other proofs.

utility rankings *and their two disagreements cancelled out.* Since the intuitively undesirable (and perhaps unacceptable) application is also needed for the argument, a suggestion presents itself to get away from the unpleasant conclusion - i.e., to make unanimous comparisons of prospects binding only in those situations in which individuals happen to agree in their judgments of the relevant probabilities. One may submit that only with such a restriction can the *ex ante* principle be reconciled with the *ex post* one, when subjective probability assignments do not coincide.[11]

5 Appendix

To make the paper self-contained, we summarize here the definitions and several facts that are needed for the proof of Theorem 1. The main tool in Gorman's (1968) [7] separability theorem, which we restate below in an adapted version.

We say that two real-valued functions defined on the same domain, f and f^*, are *ordinally equivalent*, if there exists an increasing transformation ψ such that $f^* = \psi \circ f$. Notice that if the domain of f is a connected set and f is increasing and continuous, ψ is continuous iff f^* is; it follows in particular that under the assumptions made on f, f^{-1} is also continuous.

Suppose that X_h are non-degenerate intervals in R, $h = 1, ..., l$, and $f : \times_{h=1,...,l} X_h \to R$, $f = f(x_1, ..., x_l)$. The order of the components X_h does not matter, so we will keep the same symbol f even if the order is changed. For any subset of components $I \subset \{1, ..., l\} = H$, define x_I to be the restriction of $(x_1, ..., x_l)$ to the components in I, and x_{-I} to be the retriction to the remaining components. Define the subset I to be *strictly separable* in f if the ordering of the x_I defined by $f(., \overline{\overline{x_{-I}}})$ is independent of the chosen $\overline{\overline{x_{-I}}}$, or equivalently, if there exist functions g and h such that for all $(x_1, ..., x_l)$,

$$f(x_1, ..., x_l) = g(h(x_I), x_{-I}),$$

g being increasing in its first argument. Define I to be *strictly essential* if the ordering on the x_I defined by $f(., \overline{\overline{x_{-I}}})$ is never trivial whatever the chosen $\overline{\overline{x_{-I}}}$, i.e., if for every $\overline{\overline{x_{-I}}}$, there are x_I^* and x_I^{**} such that

$$f(x_I^*, \overline{\overline{x_{-I}}}) > f(x_I^{**}, \overline{\overline{x_{-I}}}).$$

[11]This solution has recently been pursued by Gilboa, Samet, and Schmeidler (2001).

Here is a known fact about separability. When f is continuous on $\times_{h=1,...,l}X_h$, then g and h in the the defining equation of separability can also be chosen to be continuous.[12]

The following version of Gorman's theorem is weaker than the original but sufficient for the purpose.[13]

Theorem 3 *(Gorman). Suppose that f is continuous on $\times_{h=1,...,l}X_h$, with $l \geq 4$ and each X_h being strictly essential. Suppose that I^r, $I^s \subset H$ are two subsets of components that are strictly separable in f and have the following ("overlapping") properties:*

$$I^r \cap I^s,\ I^r \setminus I^s,\ I^s \setminus I^r \neq \emptyset.$$

Then, there exist continuous and increasing functions g, f^1, f^2, f^3 such that

$$f(x_1, ..., x_l) = g(f^1(x_{I^r \setminus I^s}) + f^2(x_{I^r \cap I^s}) + f^3(x_{I^s \setminus I^r}), x_{H \setminus (I^r \cup I^s)}).$$

Proof of Theorem 1. STEP 1. For given $\mathbf{p}$, φ is a function of the mn variables u^i_j, which we may rewrite as $\varphi(., \mathbf{p})$. It is a continuous function because of, say, Assumptions 5 and 7. Arranging the variables into a matrix

$$\left[u^i_j\right]^{i=1,...,n}_{j=1,...,m}$$

we see that Assumption 5 implies that each row j is separable, and Assumption 6 that each column i is separable, in $\varphi(., \mathbf{p})$. These are overlapping sets. Assumptions 1 and 4 provide a suitable domain for Gorman's theorem, and the strict essentiality condition is met because $\varphi(., \mathbf{p})$ is increasing in each u^i_j (from, say, Assumptions 5 and 7). We conclude that $\varphi(., \mathbf{p})$ can be written as an additively separable function, and in particular that:

$$V^0(\mathbf{p}, \mathbf{u}^0) = \overset{*}{V}{}^0 \left(\sum_{i=1}^{n} \sum_{j=1}^{m} \sigma^i_j(\mathbf{p}, u^i_j), \mathbf{p} \right), \tag{19}$$

[12] See, e.g., Blackorby, Primont and Russell (1978, Theorem 3.3 a) [3].

[13] Gorman (1968, Theorem 1) weakens the stringent assumption made here that the components are all strictly essential. For a reexposition, see Blackorby, Primont and Russell (1978, Theorem 4.7), and for a developped technical treatment, see von Stengel (1993, Theorem 21, p. 367) [21], who repairs the shortcomings of Gorman's initial proof.

for some $\overset{*}{V}{}^{0}$ that is continuous and increasing in its first argument, and some set of σ^i_j, $i \in N$, $j \in M$, that are continuous and increasing in their second argument.

Equation (19) implies that for any $j \in M$, $\sum_{i=1}^n \sigma^i_j(\mathbf{p}, u^i_j)$ is ordinally equivalent to $u^0_j = U^0(u^i_j, \ldots, u^n_j)$ and, accordingly, that there exists F_j such that

$$F_j\left(\sum_{i=1}^n \sigma^i_j(\mathbf{p}, u^i_j), \mathbf{p}\right) = U^0(u^1_j, \ldots, u^n_j), \tag{20}$$

F_j being increasing in its first argument. Also, given the continuity of U^0 and $\sum_{i=1}^n \sigma^i_j(\mathbf{p}, .)$,and their connected domain, F_j is continuous. In this equation, $\mathbf{p}$ may be set equal to some arbitrary probability vector $\mathbf{p}^*$. Defining $\bar{\sigma}^i_j(\cdot) = \sigma^i_j(\mathbf{p}^*, \cdot)$ and $\bar{F}_j(\cdot) = F_j(\cdot, \mathbf{p}^*)$, we have from (20) that:

$$\bar{F}_j\left(\sum_{i=1}^n \bar{\sigma}^i_j(u^i_j)\right) = U^0(u^1_j, \ldots, u^n_j),\ j \in M, \tag{21}$$

and hence

$$F_j\left(\sum_{i=1}^n \sigma^i_j(\mathbf{p}, u^i_j), \mathbf{p}\right) = \bar{F}_j\left(\sum_{i=1}^n \bar{\sigma}^i_j(u^i_j)\right),\ j \in M. \tag{22}$$

Introducing the variables $z^i_j = \bar{\sigma}^i_j(u^i_j)$ and the new functions:

$$\hat{\sigma}^i_j(\mathbf{p}, z^i_j) = \sigma^i_j(\mathbf{p}, \bar{\sigma}^{i^{-1}}_j(z^i_j)) = \sigma^i_j(\mathbf{p}, u^i_j),$$

we get

$$F_j\left(\sum_{i=1}^n \hat{\sigma}^i_j(\mathbf{p}, z^i_j),\ \mathbf{p}\right) = \bar{F}_j\left(\sum_{i=1}^n z^i_j\right),\ j \in M,$$

and finally,

$$\sum_{i=1}^n \hat{\sigma}^i_j(\mathbf{p}, z^i_j) = G_j\left(\sum_{i=1}^n z^i_j, \mathbf{p}\right),\ j \in M \tag{23}$$

for some function G_j that is continuous and increasing in its first argument. These are Cauchy equations in z^i_j, $i \in N$. The $\hat{\sigma}^i_j(\cdot)$ are continuous and increasing, which implies that the z^i_j vary over non-degenerate intervals. Following the classic functional equation theorem,[14] the solution for (23) is, for all $j \in M$:

$$\hat{\sigma}^i_j(\mathbf{p}, z^i_j) = a_j(\mathbf{p})z^i_j + b^i_j(\mathbf{p}),$$

[14]See, e.g., Eichorn (1978, Theorem 2.6.3, p.39) [4].

which implies

$$\sigma_j^i(\mathbf{p}, u_j^i) = a_j(\mathbf{p})\bar{\sigma}_j^i(u_j^i) + b_j^i(\mathbf{p}), \tag{24}$$

with $a_j(\mathbf{p})$ positive and continuous since $\hat{\sigma}_j^i$ and $\bar{\sigma}_j^i$ are increasing and continuous. Given the M equations just obtained, (19) can be rewritten as

$$V^0(\mathbf{p}, \mathbf{u}^0) = \overset{*}{V}^0 \left(\sum_{i=1}^{n} \sum_{j=1}^{m} \left(a_j(\mathbf{p})\bar{\sigma}_j^i(u_j^i) + b_j^i(\mathbf{p}) \right), \mathbf{p} \right). \tag{25}$$

STEP 2. Returning to the M equations (21), we see that the left-hand sides must be independent of j, since the right-hand sides are. Selecting the first equation, defining $\bar{\sigma}_1^i = g^i$, $i \in N$, $\bar{F}_1 = g^0$, and recalling the fact that the functions so defined are increasing and continuous, we have reached the second conclusion of the theorem:

$$U^0(u_j^1, ..., u_j^n) = g^0(\sum_{i \in N} g^i(u_j^i)), \; j \in M.$$

From the domain Assumptions 1 and 4, we can rewrite (21) as:

$$\bar{F}_j \left(\sum_{i=1}^{n} \bar{\sigma}_j^i(x^i) \right) = g^0 \left(\sum_{i=1}^{n} g^i(x^i) \right), \; j \in M, \tag{26}$$

where $x^i \in \mathcal{U}^i$, $i \in N$, is a state-independent variable. We rewrite (26) as Pexider equations:

$$\sum_{i=1}^{n} \hat{\sigma}_j^i(y^i) = \bar{F}_j^{-1} \circ g^0 \left(\sum_{i=1}^{n} y^i \right), \; j \in M, \tag{27}$$

where $\hat{\sigma}_j^i = \bar{\sigma}_j^i \circ (g^i)^{-1}$ and the new variable $y^i = g^i(u^i)$ varies over $g^i(\mathcal{U}^i)$, a non-degenerate interval from the properties of g^i. From the functional equation theorem already used, the solutions to (27) are given by:

$$\hat{\sigma}_j^i(y^i) = c_j y^i + d_j^i, \; j \in M,$$

which implies that:

$$\bar{\sigma}_j^i(u^i) = c_j g^i(u^i) + d_j^i, \; j \in M, \tag{28}$$

where $c_j > 0$, $d_j^i \in R$. (The sign of c_j follows from the fact that the $\bar{\sigma}_j^i$ and g^i are increasing; note that $c_1 = 1, d_1^i = 0, i \in N$.)

Without loss of generality, c_j and d_j^i can be absorbed into the functions a_j and b_j^i respectively, and equation (25) can be rewritten as

$$V^0(\mathbf{p}, \mathbf{u}^0) = \overset{*}{V}^0 \left(\sum_{i=1}^{n} \sum_{j=1}^{m} \left(a_j(\mathbf{p}) g^i(u_j^i) + b_j^i(\mathbf{p}) \right), \mathbf{p} \right). \tag{29}$$

STEP 3. Combining equations (29) and the consistency condition $(*)$, we observe that $V^i(\mathbf{p}, \mathbf{u}^i)$ is ordinally equivalent to $\sum_j \left(a_j(\mathbf{p}) g^i(u_j^i) + b_j^i(\mathbf{p}) \right)$. That is to say, for all $i \in N$, there is $\overset{*}{V}^i$, increasing in its first argument, such that

$$V^i(\mathbf{p}, \mathbf{u}^i) = \overset{*}{V}^i \left(\sum_{j=1}^{m} \left(a_j(\mathbf{p}) g^i(u_j^i) + b_j^i(\mathbf{p}) \right), \mathbf{p} \right). \tag{30}$$

Now, take any vector $(\mathbf{p}, \mathbf{u}^i) = (\mathbf{p}, u_1^i, \ldots, u_m^i)$ and its corresponding image by V^i:

$$V^i(\mathbf{p}, \mathbf{u}^i) = v^i.$$

From Assumptions 2.1 and 2.2:

$$V^i(\mathbf{p}, u_1^i, \ldots, u_m^i) = V^i(\mathbf{p}, v^i, \ldots, v^i).$$

Applying (30) to each side of this equation gives:

$$\overset{*}{V}^i \left(\sum_{j=1}^{m} a_j(\mathbf{p}) g^i(u_j^i) + b_j^i(\mathbf{p}), \mathbf{p} \right) = \overset{*}{V}^i \left(\sum_{j=1}^{m} a_i(\mathbf{p}) g^i(v^i) + b_j^i(\mathbf{p}), \mathbf{p} \right). \tag{31}$$

Since $\overset{*}{V}^i$ is increasing in its first argument, and g^i is increasing,

$$v^i = \left(g^i \right)^{-1} \left[\frac{\sum_{j=1}^{m} a_j(\mathbf{p}) g^i(u_j^i)}{\sum_{j=1}^{m} a_j(\mathbf{p})} \right].$$

Relabelling the coefficients $a_j(\mathbf{p}) / \sum_{k=1}^{m} a_k(\mathbf{p})$ of $g^i(u_j^i)$, we conclude that for all $i \in N$ and $\mathbf{u}^i \in \times_{j=1}^{m} \mathcal{U}^i$, and all $\mathbf{p} \in \mathcal{S}_1^M$,

$$V^i(\mathbf{p}, \mathbf{u}^i) = \left(g^i \right)^{-1} \left(\sum_{j=1}^{m} a_j(\mathbf{p}) g^i(u_j^i) \right), \tag{32}$$

which is the first conclusion of the theorem.

STEP 4. For given $\mathbf{p}$, take any $\mathbf{u}^i$ and their images $v^i = V^i(\mathbf{p}, \mathbf{u}^i)$, $i \in N$. Assumptions 2.1 and 2.2 together with the consistency condition $(*)$ imply that:

$$\begin{aligned} W(v^1, \ldots, v^n) &= W(V^1(\mathbf{p}, v^1, \ldots, v^1), \ldots, V^n(\mathbf{p}, v^n, \ldots, v^n)) \quad (33) \\ &= V^0(\mathbf{p}, U^0(v^1, \ldots, v^n), \ldots, U^0(v^1, \ldots, v^n)). \end{aligned}$$

Comparing (33) with (29), we get:

$$W(v^1, \ldots, v^n) = \overset{*}{V}{}^0 \left(\sum_{j=1}^m a_j(\mathbf{p}) \left[\sum_{i=1}^n g^i(v^i) \right] + \sum_{j=1}^m \sum_{i=1}^n b_j^i(\mathbf{p}), \mathbf{p} \right). \qquad (34)$$

Given that $\overset{*}{V}{}^0$ is increasing in its first argument and $a_j(\mathbf{p}) > 0$, $W(v^1, \ldots, v^n)$ and $\sum_{i=1}^n g^i(v^i)$ must be ordinally equivalent. That is to say, there exists w^0 such that:

$$W(v^1, \ldots, v^n) = w^0(\sum_{i=1}^n g^i(v^i), \mathbf{p}).$$

Now, the equation holds for all $v^i \in \mathcal{U}^i$ and $\mathbf{p} \in \mathcal{S}_1^M$. The left-hand side is independent of $\mathbf{p}$, so the right-hand side must also be, and the equation can be rewritten:

$$W(v^1, \ldots, v^n) = w^0(\sum_{i=1}^n g^i(v^i)).$$

Given the continuity of W and $\sum g^i$, and their connected domain (Assumptions 2.1 and 2.2), w^0 is continuous, so we have just obtained the third conclusion of the theorem.

This conclusion with $(*)$ implies that:

$$\begin{aligned} V^0(\mathbf{p}, U^0(u_1^1, ..., u_1^n), ..., U^0(u_m^1, ..., u_m^n)) &= w^0(\sum_{i=1}^n g^i(V^i(\mathbf{p}, \mathbf{u}^i)) \\ &= w^0(\sum_{i=1}^n g^i \circ g^{i\ -1} \left(\sum_{j=1}^m a_j(\mathbf{p}) g^i(u_j^i) \right)) \\ &= w^0(\sum_{j=1}^m a_j(\mathbf{p}) \left[\sum_{i=1}^n g^i(u_j^i) \right]) \\ &= w^0(\sum_{j=1}^m a_j(\mathbf{p}) g^{0-1}(U^0(u_j^1, \ldots, u_j^n)). \end{aligned}$$

Putting $v^0 = w^0 \circ u^0$, a continuous and increasing function, we have just obtained the fourth desired result.

References

[1] Ben-Porath, E., I. Gilboa, and D. Schmeidler (1997), "On the Measurement of Inequality of Uncertainty", *Journal of Economic Theory*, 75, p. 194-204.

[2] Blackorby, C., D. Donaldson, and J. Weymark (1999), Harsanyi's Social Aggregation Theorem for State-Contingent Alternatives, *Journal of Mathematical Economics*, 32, 365-387.

[3] Blackorby, C., D. Primont, and R. Russell (1978), *Duality, Separability and Functional Structure: Theory and Economic Applications*, Amsterdam.

[4] Eichorn, W. (1978), *Functional Equations in Economics*, London, Addison-Wesley.

[5] Epstein, L.G. and U. Segal (1992), "Quadratic Social Welfare Functions", *Journal of Political Economy*, 100, p. 691-712.

[6] Gilboa, I., D. Samet, and D. Schmeidler (2001), "Utilitarian Aggregation of Beliefs and Tastes", Manuscript, Tel Aviv University, December.

[7] Gorman, W. M. (1968), The Structure of Utility Functions", *Review of Economic Studies*, 35, 369-390.

[8] Gorman, W. M. (1987), "Separability", in *The New Palgrave: A Dictionary of Economics* and in C. Blackorby and T. Shorrocks (eds), *Separability and Aggregation, Collected Works of W.M. Gorman*, I, Oxford, Clarendon Press, 3-17.

[9] Hammond, P. J. (1981). "Ex-ante, and Ex-post, Welfare Optimality Under Uncertainty". *Economica*, 48, 235-250.

[10] Hammond, P. J. (1982). "Utilitarianism, Uncertainty and Information". In A.K. Sen, and B. Williams, eds, *Utilitarianism and Beyond*, Cambridge,Cambridge University Press.

[11] Hammond, P. (1988), "Consequentialist Foundations for Expected Utility", *Theory and Decision*, 25, 25-78.

[12] Karni, E. and Z. Safra (2002), "Individual Sense of Justice: A Utility Representation", *Econometrica*, 70, p. 263-284.

[13] Harsanyi, J. C. (1955). "Cardinal Welfare, Individualistic Ethics, and Interpersonal Comparisons of Utility". *Journal of Political Economy*, 63, 309-321.

[14] Lucas, R. (1972), "Expectations and the Neutrality of Money", *Journal of Economic Theory*, 4, 103-124.

[15] Machina, M. (1983), "The Economic Theory of Individual Behavior Toward Risk: Theory, Evidence, and New Directions", Stanford Institute for Mathematical Studies in the Social Sciences Technical Report 433.

[16] Machina, M. (1991), "Dynamic Consistency and Non-Expected Utility", in M. Bacharach and S. Hurley, eds, *Foundations of Decision Theory*, Oxford, Blackwell, 39-91.

[17] Mongin, P. (1995). "Consistent Bayesian Aggregation". *Journal of Economic Theory*; 66, 131-351.

[18] Mongin, P. and C. d'Aspremont (1998), "Utility Theory and Ethics", in S. Barberà, P. Hammond, and C. Seidl, eds, *Handbook of Utility Theory*, 1, New York, Kluwer, 373-482.

[19] Myerson, R.G. (1981), "Utilitarianism, Egalitarianism, and the Timing Effect in Social Choice Problems", *Econometrica*, 49, 883-897.

[20] Savage, L. J. (1972), *The Foundations of Statistics*. Wiley, New York, 2nd revised.

[21] von Stengel, B. (1993), "Independence Concepts for Continuous Utilities", *Mathematics of Operations Research*, 18, 346-389.